**Backpaper - Analysis of Several Variables (2023-24)**
**Time: 3 hours.**
*Attempt all questions, giving proper explanations.*
*You may quote any result proved in class without proof, but not results on differential forms.*

1. Give an example of a function $f : \mathbf{R}^2 \to \mathbf{R}$ that is not continuous at the origin but is continuous along every straight line through the origin. **[4 marks]**

2. Evaluate the directional derivative of $f(x, y, z) = x^2 + y^2 - z^2$ at $(3, 4, 5)$ along the curve of intersection of the two surfaces $2x^2 + 2y^2 - z^2 = 25$ and $x^2 + y^2 = z^2$. **[3 marks]**

3. Find the points on the curve of intersection of the two surfaces
$$x^2 - xy + y^2 - z^2 = 1 \text{ and } x^2 + y^2 = 1$$
which are nearest to the origin. **[4 marks]**

4. Compute $\int_C (x^2 - 2xy)dx + (y^2 - 2xy)dy$, where $C$ is a path from $(-2, 4)$ to $(1, 1)$ along the parabola $y = x^2$. **[4 marks]**

5. Consider $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ given by $\mathbf{f}(x, y) = (x, y)$. Let $g(x, y) = \int_{C_1} \mathbf{f} \cdot d\boldsymbol{\alpha} + \int_{C_2} \mathbf{f} \cdot d\boldsymbol{\beta}$ where $\boldsymbol{\alpha}$ is the parametrisation of the straight line segment $C_1$ from $(0, 0)$ to $(x, 0)$, and $\boldsymbol{\beta}$ is the parametrisation of the straight line segment $C_2$ from $(x, 0)$ to $(x, y)$. Find the gradient of $g$. **[4 marks]**

6. State Green's theorem. **[3 marks]**

7. Assume $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable at each point of the ball $B(\mathbf{a}; \epsilon)$. If $f'(\mathbf{x}; \mathbf{y}) = 0$ for $n$ independent vectors $\mathbf{y}_1, \cdots, \mathbf{y}_n$ and for every $\mathbf{x}$ in $B(\mathbf{a}; \epsilon)$, prove that $f$ is constant on $B(\mathbf{a}; \epsilon)$. **[4 marks]**

8. Consider two bounded regions $S, T \subset \mathbf{R}^2$. Denote points in $T$ by $(u, v)$ and points in $S$ by $(x, y)$. Suppose there is a 1-1 map from $T$ onto $S$ given by $x = X(u, v)$, $y = Y(u, v)$. The change of variables formula gives
$$\iint_S f(x, y)\, dxdy = \iint_T f\left[X(u, v), Y(u, v)\right] |J(u, v)| dudv,$$
for continuous functions $f$ on $S$. Derive this formula *non-rigourously* by looking at small rectangular regions in $T$; in particular show how the factor $|J(u, v)|$ arises in the right hand side. (Do NOT give the rigourous proof). **[3 marks]**

9. Show that the vector field
$$\mathbf{f}(x, y) = [\sin(xy) + xy\cos(xy)]\,\mathbf{i} + \left[x^2\cos(xy)\right]\mathbf{j}$$
on $\mathbf{R}^2$ is the gradient of a scalar field, and find a corresponding potential function $\varphi$. **[4 marks]**

10. Consider the surface in $\mathbf{r} : [0, 1]^2 \to \mathbf{R}^3$ given by $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + 4v^2\mathbf{k}$.
    (a) Let $\boldsymbol{\alpha}(t)$ be a curve in $[0, 1]^2$ so that $\mathbf{r}(\boldsymbol{\alpha}(t))$ is a curve on the surface. Show that the fundamental vector product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is normal to this curve. **[3 marks]**
    (b) Find the tangent plane to the surface at the point corresponding to $(u, v) = (0.5, 0.5)$. **[3 marks]**

11. Let $f$ be a function defined on the rectangle $Q = [a, b] \times [c, d]$. Prove or disprove:
    (a) Prove or disprove: If $|f|$ is integrable on Q then $f$ is integrable on $Q$. **[3 marks]**
    (b) Prove or disprove: If $f$ is integrable on $Q$ then $|f|$ is integrable on $Q$. **[3 marks]**

12. Find
$$\oint_C \frac{-ydx + xdy}{x^2 + y^2}$$
where $C$ is a closed curve in $\mathbf{R}^2$ which goes once around the origin in the counterclockwise direction. **[5 marks]**